2.10: SIR epidemic model

Wednesday, February 10, 2021 11:56 AM

Objectives:
1) Apply difference equations to modelling epidemics
2) Understand how to construct a model.
3) See what can go wrong with modelling.

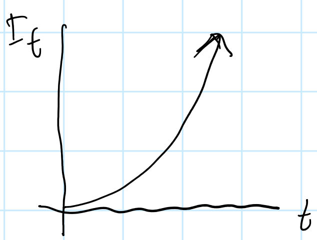
Assumption 1a: People randomly have contact events.
Assumption 1b: When an infected individual contacts a susceptible person, there is some chance of infection.

Assumption 1: Each infected person infects β other individuals in a time step.

Model 1: (Exp. growth) $I_t = I_t + \beta I_t$, where I_t = # infected at time t.

$$I_{t+1} = (1+\beta) I_t$$

$$\Rightarrow I_t = (1+\beta)^t I_0, \text{ so exponential}$$



Assumption 2: There is a total population size $N = I_t + S_t$, where S is the number of susceptible individuals.

Modified Assumption 1: β contacts per individual, but infected-infected contacts have no effect, so if contacts are random, then we get $\frac{S}{N}$ prob. of infection per contact.

Model 2: $N = I_t + S_t \Rightarrow S_t = N - I_t$

$$I_t = I_t + \frac{\beta I_t S}{N}$$

monotonically grows

$$S_t = S_t - \frac{\beta I_t S}{N}$$

$$I_t = I_t + \frac{\beta I_t (N - I_t)}{N}$$

$$I_t = I_t + \beta I_t - \frac{\beta I_t^2}{N}$$

$$I_t = (1+\beta) I_t \left(1 - \frac{\beta I_t}{N(1+\beta)}\right)$$

Suppose $I_t = N-1$

Then $I_t = N - 1 + \frac{\beta(N-1)}{N}$

If $\beta > 3$, then $I_{t+1} > N$, clearly impossible.

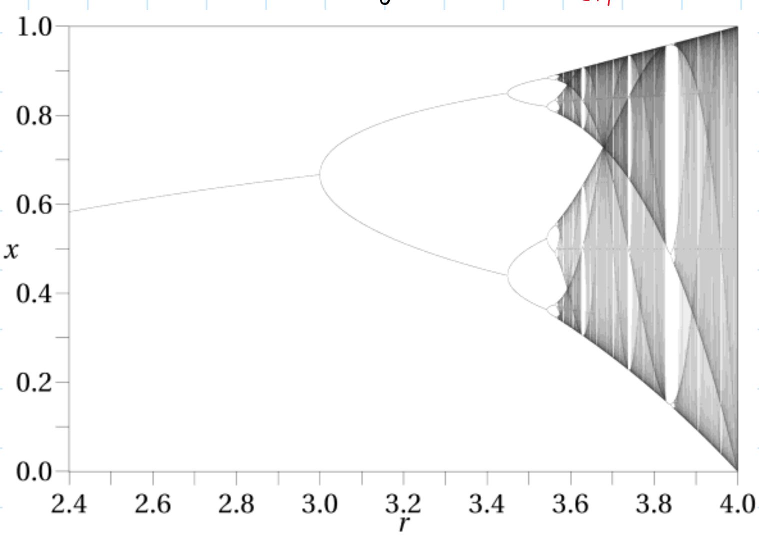
Problem is assuming $\frac{\beta S}{N}$ infections per individual.

Change variables: $x_t = \frac{\beta}{N(1+\beta)} I_t, \Rightarrow I_t = \frac{N(1+\beta)}{\beta} \cdot x_t$

$$\Rightarrow \frac{N(1+\beta)}{\beta} \cdot x_t = (1+\beta) \cdot \frac{N(1+\beta)}{\beta} \cdot x_t (1 - x_t)$$

$$\Rightarrow \underbrace{x_t = \underbrace{(1+\beta)}_{r} \cdot x_t (1 - x_t)}_{\text{discrete logistic equation}}$$

Bifurcation diagram for $x_{t+1} = r x_t (1 - x_t)$



Wikipedia for discrete logistic eq.

$\underbrace{\qquad\qquad}_{\text{cycles}}$ $\underbrace{\qquad\qquad}_{\text{chaotic behavior}}$

Assumption 3: Individuals recover with prob. γ. (No deaths)

Basic SIR model:

$$\boxed{S} \longrightarrow \boxed{I} \xrightarrow{\gamma} \boxed{R}$$

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$$S \xrightarrow[\;N\;]{\beta I} I \xrightarrow{\gamma} R$$

$$\begin{cases} S_{t+1} = S_t - \frac{\beta}{N} \cdot I_t \cdot S_t \\ I_{t+1} = I_t(1-\gamma) + \frac{\beta}{N} \cdot I_t \cdot S_t \\ R_{t+1} = R_t + \gamma I_t \end{cases}$$

Note: $N = S_t + I_t + R_t$, so $R_t = N - S_t - I_t$

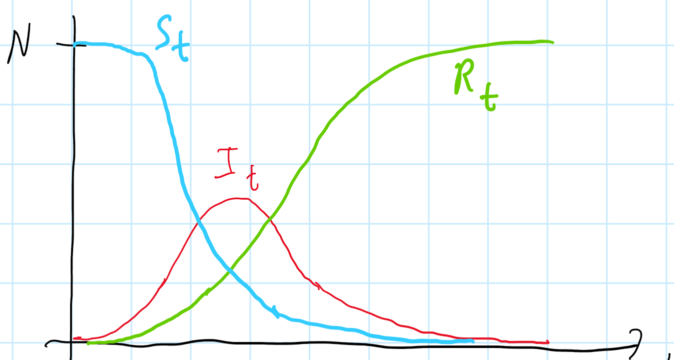
$$\begin{cases} S_{t+1} = S_t - \frac{\beta}{N} I_t \cdot S_t \\ I_{t+1} = I_t(1-\gamma) + \frac{\beta}{N} I_t \cdot S_t \end{cases}$$

Solving for equilibria:

$$\begin{cases} \bar{S} = \bar{S} - \frac{\beta}{N} \cdot \bar{I}\bar{S} \\ \bar{I} = \bar{I}(1-\gamma) + \frac{\beta}{N}\bar{I}\bar{S} \end{cases}$$

$$\begin{cases} 0 = -\frac{\beta}{N} \cdot \bar{I}\bar{S} \\ \gamma \bar{I} = \frac{\beta}{N} \cdot \bar{I}\bar{S} \end{cases}$$

$\Rightarrow \bar{I} = 0$, no constraints on $\bar{S}$, and $\bar{R} = N - \bar{S}$.



Example trajectory.

(vital dynamics) (childhood diseases like measles, chicken pox, etc.)

Assumption 4: Individuals die / are "reborn" with prob. b, and are born susceptible. (weird assumption allows constant

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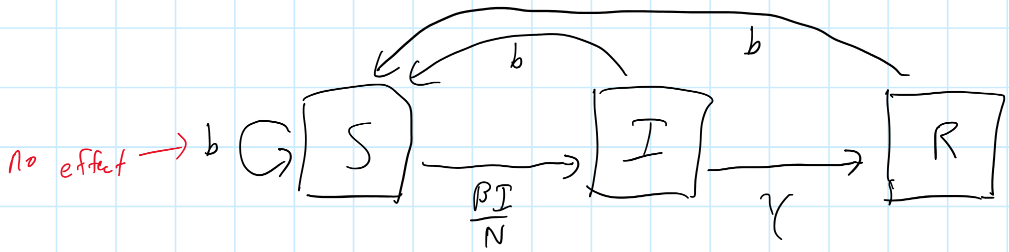

$$\begin{cases} S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + b(I_t + R_t) \\ I_{t+1} = I_t(1-\gamma-b) + \frac{\beta}{N} I_t S_t \\ R_{t+1} = R_t(1-b) + \gamma I_t \end{cases}$$

Use $R = N - S_t - I_t$:

$$\begin{cases} S_{t+1} = S_t - \frac{\beta}{N} I_t S_t + b(N - S_t) \\ I_{t+1} = I_t(1-\gamma-b) + \frac{\beta}{N} I_t S_t \end{cases}$$

Solve for equilibria

(1) $\bar{S} = N, \quad \bar{I} = 0$ — disease-free equilibrium

(2) $\bar{S} = \dfrac{N(\gamma+b)}{\beta}, \quad \bar{I} = bN\left[\dfrac{\beta - (\gamma+b)}{\beta(\gamma+b)}\right]$ — endemic equilibrium — pos. iff $\beta > \gamma + b$

Jacobian:

$$J(S,I) = \begin{bmatrix} 1 - b - \frac{\beta}{N}\cdot I & -\frac{\beta}{N}\cdot S \\ \frac{\beta}{N}\cdot I & 1-\gamma-b+\frac{\beta}{N}\cdot S \end{bmatrix}$$

At the disease free equilibrium:

$$J(N,0) = \begin{bmatrix} 1-b & -\beta \\ 0 & 1-\gamma-b+\beta \end{bmatrix}, \quad \text{so} \quad \lambda_1 = 1-b, \quad \lambda_2 = 1-\gamma-b+\beta$$

Assume $0<b<1$, so locally asymptotically stable if $\beta < \gamma + b$,

Assume $0<b<1$, so locally asymptotically stable if $\beta < \gamma + b$,
or equivalently, if $\frac{\beta}{\gamma+b} < 1$.

Define: The basic reproduction number $R_0 = \frac{\beta}{\gamma+b}$.

Note: At each time step, prob $\gamma+b$ of infected either recovering or dying, so in expectation, they will remain infectious for $\frac{1}{\gamma+b}$ time.

On average, infect β individuals each time step if $\frac{S}{N} \sim 1$.

$R_0 = \frac{\beta}{\gamma+b}$ is the expected # of new infections caused by an infected individual if $\frac{S}{N} \sim 1$.

If $R_0 > 1$, disease-free equilibrium is unstable.
$R_0 < 1$, disease-free equilibrium is locally asymptotically stable.

Ex. (Table 2.2)

Smallpox	$R_0 \sim 3-5$
Measles	$R_0 \sim 12-13$
Chickenpox	$R_0 \sim 9-10$

Analysis of endemic equilibrium $\bar{S} = \frac{N(\gamma+b)}{\beta}$, $\bar{I} = bN\left[\frac{\beta-(\gamma+b)}{\beta(\gamma+b)}\right]$

$$J(\bar{S}, \bar{I}) = \begin{bmatrix} 1-bR_0 & -\frac{\beta}{R_0} \\ b(R_0-1) & 1 \end{bmatrix}, \quad \text{where } R_0 > 1.$$

For simplicity, assume $Tr(J) = 2 - bR_0 \geq 0$.
Also, $\det(J) = 1 - bR_0 + \beta b\left(1 - \frac{1}{R_0}\right)$.

Then the Thm 2.10 (Jury conditions) are as follows:
$$2 - bR_0 < 2 - bR_0 + \beta b\left(1 - \frac{1}{R_0}\right) < 2$$

[illegible] (Jury conditions) are as

$$2 - bR_0 < 2 - bR_0 + \beta b\left(1 - \frac{1}{R_0}\right) < 2$$

$$\Longleftrightarrow \qquad 0 < \beta b\left(1 - \frac{1}{R_0}\right) < bR_0.$$

But $R_0 > 1 > \beta\left(1 - \frac{1}{R_0}\right)$, so Jury conditions satisfied.

$\Rightarrow$ The endemic eq. exists and is locally asymptotically stable if $R_0 > 1$ and $R_0 \leq \frac{2}{b}$.